

# Most edge-orderings of $K_n$ have maximal altitude

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May 25, 2016

## Abstract

Suppose the edges of the complete graph on  $n$  vertices are assigned a uniformly chosen random ordering. Let  $X$  denote the corresponding number of Hamiltonian paths that are increasing in this ordering. It was shown in a recent paper by Lavrov and Loh that this quantity is non-zero with probability at least  $1/e - o(1)$ , and conjectured that  $X$  is asymptotically almost surely non-zero. In this paper, we prove their conjecture. We further prove a partial result regarding the limiting behaviour of  $X$ , suggesting that  $X/n$  is log-normal in the limit as  $n \rightarrow \infty$ . A central part of our proof is to estimate  $\mathbb{E}X^3$ . Hence, this result may be considered one of the rare applications of a “third moment method”.

## 1 Introduction

The *altitude of an edge-ordered graph*  $(G, \preceq)$  is the length of the longest monotone (self-avoiding) path in  $G$ . The *altitude of a graph*  $G$  is the smallest altitude of any edge-ordered version of  $G$ . We denote the altitude of  $G$  by  $f(G)$ .

In 1971, Chvátal and Komlós [4] proposed the problem of determining the altitude of the complete graph on  $n$  vertices,  $K_n$ . In their paper, they relayed personal communications from R. Graham and D. Kleitman that  $\Omega(\sqrt{n}) = f(K_n) \leq \left(\frac{3}{4} + o(1)\right)n$ . Two years later, Graham and Kleitman published their result [5], stating that  $\sqrt{n - 3/4} - 1/2 \leq f(K_n) \leq 3n/4$ , and conjectured that  $f(K_n)$  is “closer to the upper bound”. The constant  $3/4$  in the upper bound on  $f(K_n)$  has successively been improved, and the current best bound is  $(\frac{1}{2} + o(1))n$ , as was shown by Calderbank, Chung, and Sturtevant in 1984 [3].

In his Master’s thesis from 1973 [12], Rödl considered the altitude of graphs with given average degree. Generalizing the lower bound for the complete graph, he showed that if  $G$  has average degree  $d$ , then  $f(G) \geq (1 - o(1))\sqrt{d}$ . For sufficiently dense graphs Rödl’s result was recently improved by Milans [9] who proved that, for  $d \gg n^{2/3}(\ln n)^{4/3}$ , we have  $f(G) = \Omega\left(dn^{-1/3}(\ln n)^{-2/3}\right)$ . In particular,  $f(K_n) = \Omega\left((n/\ln n)^{2/3}\right)$ , finally making significant improvement on Graham’s and Kleitman’s lower bound after over four decades.

A number of papers have considered the altitude of further special cases of graphs. Roditty, Shoham, and Yuster [11] proved that any planar graph has altitude at most 9, and gave examples of planar graphs with altitude at least 5. Alon [1] noted that for each  $k$ , there are  $k$ -regular graphs with altitude at least  $k$ . It can be observed that if  $\Delta(G)$  denotes the maximal degree in  $G$ , then  $f(G) \leq \Delta(G) + 1$ , hence any  $k$ -regular graph has altitude at most  $k + 1$ . It was shown by Mynhardt, Burger, Clark, Falvai and Henderson [10] that there are 3-regular graphs with altitude 4, but for  $k \geq 4$  the question remains open on whether or not there are  $k$ -regular graphs with altitude  $k + 1$ . De Silva, Molla, Pfender, Retter and Tait [13] proved that the altitude of the  $d$ -dimensional hypercube lies between  $\frac{d}{\ln d}$  and  $d$ . In the same paper, they showed that for any  $\omega(n) \rightarrow \infty$  as  $n \rightarrow \infty$  we have  $f(G(n, p)) \geq (1 - o(1)) \min\left(\sqrt{n}, \frac{np}{\omega(n)\ln n}\right)$  a.a.s., where  $G(n, p)$  is the Erdős-Rényi random graph.

Katrenič and Semanišin [7] proved that the problem of deciding whether or not a given edge-ordered graph contains a monotone Hamiltonian path is NP-complete. Hence, the problem of computing the altitude of an edge-ordered graph is NP-hard. Although it seems likely to be true, it remains an open question whether the problem of computing the altitude of a non-edge-ordered graph is also NP-hard.

In a recent article [8], Lavrov and Loh considered the altitude of a uniformly chosen edge-ordering of  $K_n$ . Recall that a (self-avoiding) path in a graph  $G$  is said to be Hamiltonian if it visits all vertices. Their main result states that a.s. the altitude of such an edge-ordering is at least  $0.85n$ , and with probability at least  $\frac{1}{e} - o(1)$  the edge-ordering contains a monotone Hamiltonian path, that is, the altitude is  $n - 1$ . They consequently gave the following natural conjecture.

**Conjecture 1.1.** (*Lavrov-Loh*) *With probability tending to 1, a random edge-ordering of  $K_n$  contains a monotone Hamiltonian path.*

The aim of this paper is to prove this conjecture.

Consider a uniformly chosen random edge-ordering of  $K_n$ . Let  $X = X_n$  denote the number of Hamiltonian paths that are increasing with respect to this ordering. As there are  $n!$  Hamiltonian paths in  $K_n$ , and each path is increasing with probability  $\frac{1}{(n-1)!}$ , we have  $\mathbb{E}X = n$ . Lavrov and Loh gave an elegant argument for estimating the second moment of  $X$ , yielding  $\mathbb{E}X^2 \sim en^2$ . Their result that  $\mathbb{P}(X > 0) \geq \frac{1}{e} - o(1)$  follows immediately from this by the second moment method.

The key idea to our approach is to relate the distribution of  $X$  to its conditional distribution given the event that a certain path is increasing. As we shall see, this boils down to showing certain third moment estimates of  $X$ . Thus, one could with a bit of a stretch consider this as one of the few occurrences of the “third moment method”. We have the following result.

**Theorem 1.2.** *Asymptotically almost surely as  $n \rightarrow \infty$  we have  $X_n > 0$ . Moreover, for any  $x > 0$ ,*

$$\limsup_{n \rightarrow \infty} \mathbb{P}(X_n \leq xn) \leq e \cdot x. \quad (1.1)$$

In fact, our approach gives us a lot more information about  $X_n$  for large  $n$ . Given Theorem 1.2 it is natural to ask further whether or not  $X_n/\mathbb{E}X_n = X_n/n$ , has a limiting distribution as  $n \rightarrow \infty$ . As  $\mathbb{E}X_n^2 \sim en^2$ , the standard deviation of  $X_n/n$  is  $e - 1 + o(1)$ , so we would expect this to have a non-trivial limiting distribution. We note that, as  $X_n/n \geq 0$  and  $\mathbb{E}X_n/n = 1$ , the sequence  $\{X_n/n\}_{n=1}^\infty$  is tight, meaning that no mass of  $X_n/n$  escapes to infinity as  $n$  increases. Hence, for any sequence  $\{X_{m_i}/m_i\}_{i=1}^\infty$  with  $m_i \rightarrow \infty$  there is a subsequence  $\{X_{n_i}/n_i\}_{i=1}^\infty$  that converges in distribution. By Theorem 1.2 we know that the limit of any such sequence has no mass at 0. We note that if all such converging sequences have the same limit, then  $X_n/n$  converges to that distribution as  $n \rightarrow \infty$ .

Recall that the *log-normal* distribution with parameters  $\mu$  and  $\sigma$ ,  $\log\mathcal{N}(\mu, \sigma)$ , is the distribution of  $Y = e^Z$  where  $Z \sim \mathcal{N}(\mu, \sigma)$ .

**Proposition 1.3.** *Let  $F(x)$  denote the cumulative distribution function of the limit of any weakly converging subsequence  $\{X_{n_i}/n_i\}_{i=1}^\infty$ . We then have*

$$\int_0^\infty x^k dF(x) = e^{k(k-1)/2} \quad (1.2)$$

*for any (not necessarily positive) integer  $k$ . That is,  $F$  has the same moments as a log-normal random variable with  $\mu = -\frac{1}{2}$  and  $\sigma = 1$ . Moreover, if we let  $G(t) = F(e^t)$ , equivalently  $G(t)$  is*

the CDF of the limit of  $\ln(X_{n_i}/n_i)$ , then  $G(t)$  can be written as

$$dG(t) = e^{-(t+\frac{1}{2})^2/2} d\nu(t) \quad (1.3)$$

where  $\nu(t)$  is a 1-periodic positive measure on  $\mathbb{R}$ .

An important caveat relating to this proposition is that the log-normal distribution is *M-indeterminate*, meaning that there exist other random variables that have the same moments. Roughly speaking, a random variable  $\xi$  is M-determinate if its sequence of moments is not growing too quickly. A well-known sufficient condition is that the generating function  $\mathbb{E}e^{\lambda\xi} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}\xi^k$  converges in some interval around 0, which is not the case for the moments above. As a proof that  $\log\mathcal{N}(-\frac{1}{2}, 1)$  is M-indeterminate, one can observe that (1.3) implies (1.2), that is, for any 1-periodic positive locally finite measure  $\nu$ , normalized such that  $\int_{\mathbb{R}} dG(t) = 1$ , the corresponding distribution  $F$  has these moments. In [14] further examples of distributions with these moments are constructed, meaning that the characterization of  $F$  in (1.3) is stronger than just giving its moments.

While this proposition deals directly only with weak limit points of  $\{X_n/n\}_{n=1}^{\infty}$ , we can use it to show that the entire sequence has certain properties asymptotically – otherwise there must be a converging subsequence that asymptotically does not have this property. For example, as no  $dF$  as above has compact support, it follows that for any  $M > 0$  we must have  $\mathbb{P}(X_n/n > M)$  bounded away from 0 for large  $n$ . Since (1.2) holds for negative powers of  $x$ , we have  $F(x) \leq \int_0^{\infty} \frac{x^k}{y^k} dF(y) = e^{k(k+1)/2} x^k$  for any positive integer  $k$ . Hence we can strengthen (1.1) to

$$\limsup_{n \rightarrow \infty} \mathbb{P}(X_n \leq x n) \leq \min_{k \in \mathbb{Z}_+} e^{k(k+1)/2} x^k. \quad (1.4)$$

The proposition also more or less implies that  $\mathbb{E}X^k \sim n^k e^{k(k-1)/2}$ . More precisely, it shows that there is a truncation  $\hat{X}_n$  of  $X_n$  such that  $\mathbb{P}(X_n = \hat{X}_n) \rightarrow 1$  and  $\mathbb{E}\hat{X}^k \sim n^k e^{k(k-1)/2}$ . Moreover, if one can show that  $\mathbb{E}X^k = O(n^k)$  for each fixed positive integer  $k$ , then it follows that these moment estimates hold without truncation.

The question remains open whether or not  $X_n/n$  has a limiting distribution, and in that case which of the distributions of the form prescribed in Proposition 1.3 it is. It seems that new ideas are needed to make any further progress on this problem. Nevertheless, I believe that Proposition 1.3 provides strong evidence for the following statement.

**Conjecture 1.4.** *As  $n \rightarrow \infty$ ,  $X_n/n$  converges in distribution to a  $\log\mathcal{N}(-\frac{1}{2}, 1)$  random variable.*

In the remaining parts of the article, we will prove Theorem 1.2 and Proposition 1.3 in parallel. Section 2 gives the main idea of our approach and shows how our results can be reduced to showing third moment estimates for  $X$ . These estimates will then be derived in Section 3, completing the proof of both statements.

## 2 Proof of Theorem 1.2 and Proposition 1.3

Let  $P_0$  be any fixed Hamiltonian path in  $K_n$ . Let  $Y = Y_n$  denote the conditional number of increasing Hamiltonian paths given that  $P_0$  is increasing. Observe that if one first generates a uniformly chosen edge-ordering of  $K_n$  and then switches the positions of edges along  $P_0$  in the ordering such that  $P_0$  becomes increasing, then the resulting edge-ordering has the same distribution as a uniform ordering conditioned on  $P_0$  being increasing. Hence this gives a natural way to couple  $X$  to  $Y$ .

Let us first note a simple relation between the distributions of  $X$  and  $Y$ , which is essentially a reformulation of the definition of  $Y$ .

**Proposition 2.1.** *For any  $k \geq 0$ ,*

$$\mathbb{P}(Y = k) = \frac{k}{n} \mathbb{P}(X = k).$$

*Proof.* Note that the conditional probability that  $P_0$  is increasing given  $X = k$  is, by symmetry, proportional to  $k$ . Hence, by Bayes' theorem,  $\mathbb{P}(Y = k) = \frac{k \cdot \mathbb{P}(X=k)}{\sum_l l \cdot \mathbb{P}(X=l)}$ . Observing that  $\sum_l l \cdot \mathbb{P}(X = l) = \mathbb{E}X = n$  completes the proof.  $\square$

The key insight that allows us to prove our main result is that, in the limit as  $n \rightarrow \infty$ ,  $X$  and  $Y$  have the same distribution except that  $Y$  is scaled up by some constant factor. In other words, conditioning on one fixed path being increasing will multiply the number of increasing Hamiltonian paths by a constant factor, but otherwise not affect the distribution. In fact what we will show is that, for some suitable constant  $C_1$  and with the coupling of  $X$  and  $Y$  as indicated above, “ $Y - C_1 X$  is small for large  $n$ ”.

Heuristically, if  $Y - C_1 X$  is small, we would expect  $\mathbb{E}Y$  to be close to  $C_1 \mathbb{E}X$ . Recall that  $\mathbb{E}X = n$ . For any Hamiltonian path  $P$ , let  $X_P$  denote the indicator that  $P$  is increasing. Hence  $X = \sum_P X_P$ . Computing the expectation of  $Y$  from the definition, we get

$$\begin{aligned} \mathbb{E}Y &= \sum_A \mathbb{E}[X_A | X_{P_0} = 1] \\ &= \frac{1}{\mathbb{P}(X_{P_0} = 1)} \sum_A \mathbb{E}X_{P_0} X_A \\ &= (n-1)! \frac{1}{n!} \sum_P \sum_A \mathbb{E}X_P X_A = \frac{1}{n} \mathbb{E}X^2. \end{aligned} \tag{2.1}$$

As mentioned in the introduction, Lavrov and Loh [8] showed that the second moment of  $X$  is  $\sim en^2$ , hence we expect  $C_1 = e$ . We remark that  $\mathbb{E}Y = \frac{1}{n} \mathbb{E}X^2$  can be derived in simpler manner using Proposition 2.1, but we have chosen to show the derivation in (2.1) as this type of argument will recur below.

**Proposition 2.2.** *We have  $\mathbb{E}[(Y - eX)^2] = o(n^2)$ .*

In proving this proposition, it turns out to be useful to introduce a third random variable,  $Z = Z_n$ , the number of increasing Hamiltonian paths that are edge disjoint from  $P_0$ . We similarly expect  $Z - C_2 X$  to be small for large  $n$  and for some constant  $C_2$ . As the joint distribution of  $X$  and  $Z$  does not depend on which path  $P_0$  is chosen, we may consider  $P_0$  to be chosen uniformly at random, independently of the edge-ordering of  $K_n$ . For any two paths  $A, B$  in  $K_n$ , let  $|A \cap B|$  denote the number of edges they have in common. We have

$$\begin{aligned} \mathbb{E}[Z|X] &= \sum_A \mathbb{E}[\mathbb{1}_{|A \cap P_0|=0} X_A | X] \\ &= \sum_A \mathbb{P}(|A \cap P_0| = 0) \mathbb{E}[X_A | X] \end{aligned}$$

It is clear that  $\mathbb{P}(|A \cap P_0| = 0)$  does not depend on  $A$ . Let us label the vertices of  $K_n$  by  $1, 2, \dots, n$ , and consider the case of the path  $I$  with vertex sequence  $\{1, 2, \dots, n\}$ . Then  $\mathbb{P}(|I \cap P_0| = 0)$  is the probability that a uniformly chosen random permutation  $\{v_1, v_2, \dots, v_n\}$  of  $\{1, 2, \dots, n\}$  satisfies  $|v_{i+1} - v_i| > 1$  for all  $i$ . It was shown, although somewhat sketchily, by Wolfowitz [15] and elaborated on by Kaplansky [6] that this tends to  $e^{-2}$  as  $n \rightarrow \infty$ . A similar statement (see Claim 3.9 below) will be shown later in the paper. It follows that  $\mathbb{E}[Z|X] = \mathbb{P}(|I \cap P_0| = 0) X \sim e^{-2} X$ . Hence  $\mathbb{E}Z \sim e^{-2} \mathbb{E}X = e^{-2} n$  and so  $C_2 = e^{-2}$ .

Writing  $Y - eX = (Y - e^3Z) + (e^3Z - eX)$ , it follows that  $\mathbb{E}[(Y - eX)^2] \leq 2\mathbb{E}[(Y - e^3Z)^2] + 2\mathbb{E}[(e^3Z - eX)^2]$ . Hence, in order to prove Proposition 2.2, it suffices to show that

$$\mathbb{E}[(Y - e^3Z)^2] = \mathbb{E}Y^2 - 2e^3\mathbb{E}YZ + e^6\mathbb{E}Z^2 = o(n^2),$$

and

$$\mathbb{E}[(e^3Z - X)^2] = e^4\mathbb{E}Z^2 - 2e^2\mathbb{E}XZ + \mathbb{E}X^2 = o(n^2).$$

As already stated,  $\mathbb{E}X^2 \sim en^2$ . Similarly, by the above calculation we have  $\mathbb{E}XZ = \mathbb{E}[X\mathbb{E}[Z|X]] \sim e^{-2}\mathbb{E}X^2 \sim e^{-1}n^2$ . Furthermore, following the argument in (2.1), we have

$$\mathbb{E}Y^2 = \sum_A \sum_B \mathbb{E}[X_A X_B | X_{P_0} = 1] = \frac{1}{n} \sum_P \sum_A \sum_B \mathbb{E}[X_P X_A X_B] = \frac{1}{n} \mathbb{E}X^3, \quad (2.2)$$

so estimating the second moment of  $Y$  is equivalent to estimating the third moment of  $X$ . Similarly,  $\mathbb{E}YZ$  and  $\mathbb{E}Z^2$  can be expressed as sums over subsets of the terms in  $\mathbb{E}X^3$ . More precisely

$$\mathbb{E}YZ = \frac{1}{n} \sum_P \sum_A \sum_{|B \cap P|=0} \mathbb{E}[X_P X_A X_B],$$

and

$$\mathbb{E}Z^2 = \frac{1}{n} \sum_P \sum_{|A \cap P|=0} \sum_{|B \cap P|=0} \mathbb{E}[X_P X_A X_B].$$

Proposition 2.2 follows if one can show the following third moment estimates on  $X$ .

**Proposition 2.3.** *As  $n \rightarrow \infty$ , we have*

$$\mathbb{E}X^3 = \sum_A \sum_B \sum_C \mathbb{E}X_A X_B X_C \sim e^3 n^3, \quad (2.3)$$

and furthermore,

$$\sum_A \sum_B \sum_{|C \cap A|=0} \mathbb{E}X_A X_B X_C \sim n^3, \quad (2.4)$$

$$\sum_A \sum_{|B \cap A|=0} \sum_{|C \cap A|=0} \mathbb{E}X_A X_B X_C \sim e^{-3} n^3. \quad (2.5)$$

The proof of this proposition involves some rather involved combinatorial arguments, and will be given in Section 3.

Combining Propositions 2.1 and 2.2 we get immediate simple proofs of Theorem 1.2 and Proposition 1.3.

*Proof of Theorem 1.2.* Let  $x, \varepsilon > 0$ . Observe that if  $X \leq xn$ , then either  $Y \leq (e + \varepsilon)xn$  or  $Y - eX > \varepsilon xn$ . Hence by Proposition 2.2 we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}(X_n \leq xn) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(Y_n \leq (e + \varepsilon)xn).$$

By Proposition 2.1, we further have

$$\mathbb{P}(Y_n \leq (e + \varepsilon)xn) = \sum_{k=0}^{\lfloor (e+\varepsilon)xn \rfloor} \frac{k}{n} \mathbb{P}(X_n = k) \leq (e + \varepsilon)x \sum_{k=0}^{\infty} \mathbb{P}(X_n = k) = (e + \varepsilon)x.$$

□

*Proof of Proposition 1.3.* Consider the sequence  $Y_{n_i}/n_i$ . By Proposition 2.1 we know that

$$\mathbb{P}(Y_{n_i}/n_i \leq x) \rightarrow \int_0^x y dF(y) \text{ as } i \rightarrow \infty$$

for almost all  $x \geq 0$ . Further, by Proposition 2.2 we have

$$\mathbb{P}(Y_{n_i}/n_i \leq x) \rightarrow F(e^{-1}x) \text{ as } i \rightarrow \infty$$

for almost all  $x \geq 0$ . As a consequence of this, we have

$$x dF(x) = e^{-1} dF(e^{-1}x),$$

or equivalently, using  $dG(t) = e^t dF(e^t)$ ,

$$dG(t) = e^{-t} dG(t-1).$$

It follows immediately that if we define  $\nu$  by  $d\nu(t) = e^{(t+\frac{1}{2})^2/2} dG(t)$ , then  $d\nu(t-1) = e^{(t-\frac{1}{2})^2/2} dG(t-1) = e^{t+(t-\frac{1}{2})^2/2} dG(t) = d\nu(t)$ . Hence  $\nu$  is 1-periodic and satisfies (1.3).

As for (1.2), we have

$$\begin{aligned} \int_0^\infty x^k dF(x) &= \int_{-\infty}^\infty e^{kt} dG(t) = \int_{-\infty}^\infty e^{kt} e^{-(t+\frac{1}{2})^2/2} d\nu(t) \\ &= e^{k(k-1)/2} \int_{-\infty}^\infty e^{-(t-k+\frac{1}{2})^2/2} d\nu(t) \\ &= e^{k(k-1)/2} \int_{-\infty}^\infty e^{-(t+\frac{1}{2})^2/2} d\nu(t), \end{aligned}$$

where the integral on the last line evaluates to one as  $e^{-(t+\frac{1}{2})^2/2} d\nu(t) = dG(t)$  is a probability measure.  $\square$

### 3 Third moment analysis

In what follows, we will use capital  $A, B$  and  $C$  to denote (directed) Hamiltonian paths. For such a path  $A$ ,  $X_A$  is the indicator for the event that the edges in  $A$  are in ascending order. We denote by  $|A \cap B|$  and  $|(A \cup B) \cap C|$  etc. the number of edges of the respective sets.

As a first step in our proof, we rewrite the sums in equations (2.3), (2.4) and (2.5). Starting with the identity

$$\mathbb{E}X^3 = \sum_{A,B,C} \mathbb{E}X_A X_B X_C = \sum_{A,B,C} \mathbb{P}(A, B, C \text{ are all increasing}),$$

we note that  $\mathbb{P}(A, B, C \text{ are all increasing}) = \sum_{\preceq} \frac{1}{|A \cup B \cup C|!}$  where the sum goes over all edge-orderings  $\preceq$  of  $A \cup B \cup C$  that make  $A, B$  and  $C$  all increasing. Hence, we may equivalently write  $\mathbb{E}X^3$  as a sum over all edge-ordered path triples  $(A, B, C, \preceq)$  where all paths are increasing:

$$(2.3) = \sum_{(A,B,C,\preceq)} \frac{1}{|A \cup B \cup C|!}.$$

Similarly we get the sums

$$(2.4) = \sum_{(A,B,C,\preceq)} \frac{\mathbb{1}_{|A \cap C|=0}}{|A \cup B \cup C|!},$$

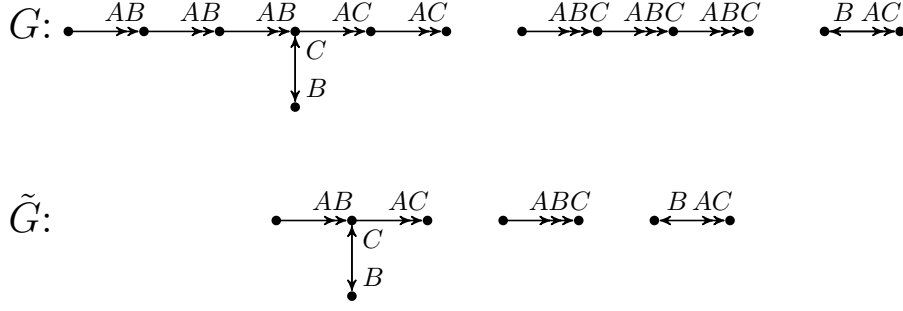


Figure 1: An example of a common edge graph  $G$  and the corresponding reduced common edge graph  $\tilde{G}$ . In both cases we consider the edges as being ordered from left to right. Note that the components of  $G$  and  $\tilde{G}$  need not be chains. In fact, they can even contain cycles, though this is not shown here.

and

$$(2.5) = \sum_{(A,B,C,\preceq)} \frac{\mathbb{1}_{|A \cap B|=0} \mathbb{1}_{|A \cap C|=0}}{|A \cup B \cup C|!}.$$

Let  $(A, B, C, \preceq)$  be a triple of Hamiltonian paths together with a corresponding edge-ordering. Below we will always assume that all three paths are increasing with respect to  $\preceq$ . For brevity, we will refer below to such a triple as an *edge-ordered triple* and suppress the  $\preceq$  in the notation.

Dealing with edge-ordered triples  $(A, B, C)$ , an important concept is the corresponding *common edge graph*, denoted below by  $G$ . This is defined as follows. Take the induced subgraph of  $K_n$  consisting of all edges that are contained in at least two of the paths (any vertex of degree zero is removed). Label each edge according to which paths the edge is contained in, and in which direction these traverse the edge. We consider  $G$  as an edge-ordered graph by letting it inherit the order of  $(A, B, C)$ . See Figure 1 for an example.

For a fixed edge-ordered triple, the common edge graph may contain paths of length at least two,  $v_0, e_1, v_1, e_2, \dots, e_l, v_l$ , where  $e_1, e_2, \dots, e_l$  are shared by a fixed subset of  $\{A, B, C\}$ . In other words, they are all contained in one of  $(A \cap B) \setminus C$ ,  $(A \cap C) \setminus B$ ,  $(B \cap C) \setminus A$  or  $A \cap B \cap C$ . Note that the edge-ordering implies that the involved paths among  $A, B$ , and  $C$  traverse  $e_1, e_2, \dots, e_l$  in a common direction.

**Claim 3.1.** *For any sequence  $v_0, e_1, v_1, e_2, \dots, e_l, v_l$  as above,  $e_1, e_2, \dots, e_l$  are next to each other in the edge-ordering of the common edge graph. Moreover, for any  $0 < i < l$ , the only edges incident to  $v_i$  are  $e_i$  and  $e_{i+1}$ .*

*Proof.* Let us, without loss of generality, assume that the edges are in  $A \cap B$ . As  $A, B$ , and  $C$  are self-avoiding, no edges in  $A$  or  $B$  can come between  $e_i$  and  $e_{i+1}$  in the edge-ordering. Hence any such edge is unique to  $C$ , and not in the common edge graph. Similarly  $e_i$  and  $e_{i+1}$  are the only edges in  $A$  and  $B$  respectively incident to  $v_i$ , which makes them the only incident edges in the common edge graph.  $\square$

Motivated by the above claim, we define the *reduced common edge graph*,  $\tilde{G}$ , of an edge-ordered triple as the graph obtained by collapsing all paths as above in the common edge graph to a single edge, preserving edge labels and the edge-ordering, see Figure 1. It follows from the

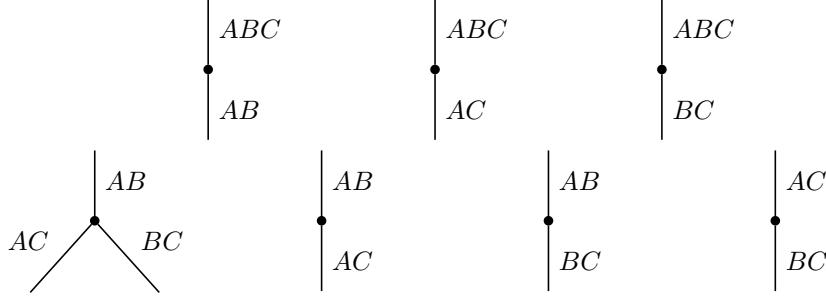


Figure 2: The possible neighborhoods of a vertex in a reduced common edge graph.

preceding claim that  $G$ , including edge-ordering, can be uniquely recovered from  $\tilde{G}$  if one knows the lengths of the collapsed chains.

For a given edge-ordered triple  $(A, B, C)$ , we let

$$k_1 = \text{the number of common segments between } A \text{ and } B, \quad (3.1)$$

$$k_2 = \text{the number of common segments between } A \cup B \text{ and } C, \quad (3.2)$$

$$k_3 = \text{the number of components in the common edge graph} \quad (3.3)$$

that do not contain an edge common to  $A$  and  $B$ ,

$$k_4 = \text{the number of vertices in the common edge graph which} \quad (3.4)$$

are incident to an edge in  $A \cap C$  and one in  $B \cap C$ , but  
none in  $A \cap B$ .

Note that each of these quantities are uniquely determined by the reduced common edge graph.

**Claim 3.2.** *Let  $k_1, k_2, k_3$  and  $k_4$  be given non-negative integers. Then (up to isomorphism) the number of reduced common edge graphs  $\tilde{G}$  such that  $k_1(\tilde{G}) = k_1, k_2(\tilde{G}) = k_2, k_3(\tilde{G}) = k_3$  and  $k_4(\tilde{G}) = k_4$  is at most  $e^{O(k_1+k_2+k_4)}$ . Moreover, any such  $\tilde{G}$  contains at most  $O(k_1 + k_2 + k_4)$  edges.*

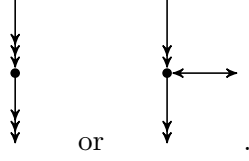
*Proof.* We first bound the number of edges in  $\tilde{G}$ . Consider the possible sets of edges incident to a vertex  $v \in \tilde{G}$ . To reduce the number of cases, we can ignore the directions in which the paths traverse the various edges, and only distinguish which paths each edge is contained in. This is illustrated in Figure 2. We see clearly that  $v$  is either the end-point of a common segment of  $A$  and  $B$  (the first three cases on the second row), in which case it contributes by  $\frac{1}{2}$  to  $k_1(\tilde{G})$ , an end-point of a common segment of  $A \cup B$  and  $C$  (the first row), in which case it contributes  $\frac{1}{2}$  to  $k_2(\tilde{G})$ , or counted in  $k_4(\tilde{G})$  (the last case on the second row). As any vertex has at most three incident edges, it follows that  $|E(\tilde{G})| = O(k_1 + k_2 + k_4)$ .

Suppose that we construct  $\tilde{G}$  by adding the edges one at a time, in ascending order. For each new edge we need to choose its label, whether or not its end-points are already in the graph, and in that case which vertices these are. Observe that if an edge  $e$  in  $\tilde{G}$  shares a vertex with some edge  $e' \preceq e$ , then  $e$  and  $e'$  must be incident edges in either  $A, B$  or  $C$ . Hence there are at most three vertices where a new edge can be attached. Thus  $\tilde{G}$  can be encoded in  $O(|E(\tilde{G})|)$  bits.  $\square$

**Claim 3.3.** *Let  $(A, B, C)$  be an edge-ordered triple in  $K_n$ . Then, for any  $\varepsilon > 0$ , either  $|(A' \cup B') \cap C'| \leq (1 - \varepsilon)n$  for some permutation  $(A', B', C')$  of  $(A, B, C)$ , or  $|A \cap B \cap C| \geq (1 - 18\varepsilon)n$ .*

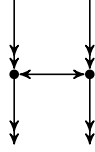


*Proof.* Let  $G$  be the common edge graph of  $(A, B, C)$ . We define the weight of a vertex in  $G$ , denoted by  $w(v)$ , as the number of edges incident to  $v$  in  $G$ , where an edge is counted with multiplicity 2 if it is shared by two paths and 3 if shared by all three. It is not too hard to see that the only ways a vertex can have the maximal weight of 6 is



Let us denote the number of these types of vertices by  $x$  and  $y$  respectively.

Now, one can observe that no  $\longleftrightarrow$  edge can have two end-points of weight 6, as it is impossible to order the edges in



such that the path segments of  $A$ ,  $B$  and  $C$  are all increasing. Furthermore, as each vertex of weight  $\leq 5$  can have at most two  $\longleftrightarrow$  edges, it follows that  $y \leq \text{number of such edges} \leq 2 \cdot (\text{number of vertices of weight} \leq 5) = 2(n - x - y)$ . In particular  $y \leq \frac{2}{3}(n - x)$ .

Now, assume that  $|A \cap B \cap C| < (1 - 18\varepsilon)n$ . One readily sees that  $|A \cap B \cap C| \geq x$ . We then have

$$\begin{aligned} & |(A \cup B) \cap C| + |(A \cup C) \cap B| + |(B \cup C) \cap A| \\ &= \frac{1}{2} \sum_{v \in G} w(v) \leq 3(x + y) + \frac{5}{2}(n - x - y) \\ &\leq \frac{17n}{6} + \frac{x}{6} \leq 3(1 - \varepsilon)n, \end{aligned}$$

where in the second to last step we used  $y \leq \frac{2}{3}(n - x)$ .  $\square$

Let  $\tilde{G}$  be a reduced common edge graph, and let  $c_{AB}, c_{AC}, c_{BC}$  and  $c_{ABC}$  be non-negative integers. To simplify notation we will write  $\bar{c} = (c_{AB}, c_{AC}, c_{BC}, c_{ABC})$ . Let  $T_n(\tilde{G}, \bar{c})$  denote the number of edge-ordered triples  $(A, B, C)$  corresponding to  $\tilde{G}$  such that

$$|(A \cap B) \setminus C| = c_{AB}, \quad |(A \cap C) \setminus B| = c_{AC}, \quad |(B \cap C) \setminus A| = c_{BC}$$

and

$$|A \cap B \cap C| = c_{ABC}.$$

We furthermore let  $t_n(\tilde{G}, \bar{c}) = T_n(\tilde{G}, \bar{c}) / (3n - 3 - c_{AB} - c_{AC} - c_{BC} - 2c_{ABC})!$ . Note that this means that  $\mathbb{E}X^3 = \sum_{\tilde{G}} \sum_{\bar{c}} t_n(\tilde{G}, \bar{c})$ .

The following proposition gives a reasonable bound for  $T_n(\tilde{G}, \bar{c})$  provided either  $c_{ABC}$  is close to  $n$ , or  $n - c_{AC} - c_{BC} - c_{ABC}$  is of order  $n$ . By Claim 3.3 we know that we can always permute the paths in an edge-ordered triple such that one of these two properties is satisfied.

**Proposition 3.4.** *For  $\tilde{G}$  and  $\bar{c}$  as above, we have*

$$\begin{aligned} T_n(\tilde{G}, \bar{c}) &\leq \left( \prod_{\alpha} \binom{c_{\alpha} - 1}{l_{\alpha} - 1} \right) \binom{2n - 2 - 2c_{AB} - 2c_{ABC} + k_1}{n - 1 - c_{AB} - c_{ABC}, n - 1 - c_{AB} - c_{ABC}, k_1} \\ &\cdot \binom{2n - 2 - 2c_{AB} - 2c_{ABC}}{k_3} \binom{3n - 3 - c_{AB} - 2c_{AC} - 2c_{BC} - 3c_{ABC} + k_2}{n - 1 - c_{AC} - c_{BC} - c_{ABC}} \\ &\cdot n!(n - c_{AB} - c_{ABC} - k_1 - k_4)!(n - c_{AC} - c_{BC} - c_{ABC} - k_2)!, \end{aligned}$$

where the product goes over  $\alpha \in \{AB, AC, BC, ABC\}$ ,  $k_i = k_i(\tilde{G})$  for  $i = 1, 2, 3, 4$  and  $l_\alpha = l_\alpha(\tilde{G})$  denotes the number of edges in  $\tilde{G}$  labelled with  $\alpha$ , that is the number of  $AB$ -,  $AC$ -,  $BC$ -, and  $ABC$ -edges respectively in  $\tilde{G}$ .

**Remark 3.5.** Note that  $c_{AB}$ ,  $c_{AC}$ ,  $c_{BC}$  and  $c_{ABC}$  are allowed to be 0. In that case, we should think of the corresponding binomial factor  $\binom{c_\alpha-1}{l_\alpha-1}$  as one if  $l_\alpha = 0$  and 0 otherwise, which is consistent with the interpretation of  $\binom{c_\alpha-1}{l_\alpha-1}$  as the number of ways to place  $c_\alpha$  indistinguishable balls in  $l_\alpha$  bins such that each bin contains at least one ball.

*Proof.* In counting the number of edge-ordered triples  $(A, B, C, \preceq)$  satisfying the conditions above, it is useful to consider what we refer to as an *edge list*, first of  $A$  and  $B$ , and then of  $A$ ,  $B$  and  $C$ . This is defined as a list of length  $|A \cup B|$  or  $|A \cup B \cup C|$  respectively where the  $i$ :th entry denotes which of  $A$ ,  $B$  and  $C$  the  $i$ :th smallest edge in  $A \cup B$  or  $A \cup B \cup C$  respectively is contained in. This serves both to encode the edge-order of  $A \cup B \cup C$  and the number of unique edges the paths have between common segments.

STEP 1: Choose the common edge graph  $G$ .

It suffices to choose the lengths of each collapsed chain in  $\tilde{G}$ . By Claim 3.1 we know that this uniquely defines the edge-ordering of  $G$ . For each  $\alpha \in \{AB, AC, BC, ABC\}$  we need to divide up  $c_\alpha$  edges between  $l_\alpha$  segments. Hence, there are at most  $\prod_\alpha \binom{c_\alpha-1}{l_\alpha-1}$  ways to do this.

STEP 2: Choose the edge list of  $(A, B)$ .

Given  $G$ , we know the number of common segments of  $A$  and  $B$  and their respective lengths. As no edge unique to  $A$  or  $B$  can occur during one of the common segments, any edge list can, given  $G$ , be encoded as a string containing  $|A \setminus B| = n-1-c_{AB}-c_{ABC}$  A:s,  $|B \setminus A| = n-1-c_{AB}-c_{ABC}$  B:s, and  $k_1$  D:s, denoting the placement in the order of the edges unique to  $A$ ,  $B$ , and the common segments of  $A$  and  $B$  respectively. Hence the number of such edge lists is at most the multinomial coefficient,  $\binom{2n-2-2c_{AB}-2c_{ABC}+k_1}{n-1-c_{AB}-c_{ABC}, n-1-c_{AB}-c_{ABC}, k_1}$ .

STEP 3: For each edge in  $G$ , locate the corresponding entry in the edge list of  $(A, B)$ .

Note that each edge in  $G$  is contained in at least one of  $A$  and  $B$ , and is hence present in the edge list. The position of any edge common to  $A$  and  $B$  is immediately determined by the edge-ordering of  $G$  – the  $i$ :th common edge between  $A$  and  $B$  in  $G$  corresponds to the  $i$ :th  $AB$  entry in the edge list. Moreover, if we know which position one edge in  $G$  has in the edge list, then we also know the position any adjacent edge in  $G$  as such an edge is either the previous or next edge in one of  $A$  and  $B$ . Thus, the position of one edge implies the positions of all edges in the same component. Hence, it suffices to choose the location of one edge from each of the  $k_3$  components in  $G$  that does not contain a common edge of  $A$  and  $B$ . As any such edge cannot be common to  $A$  and  $B$ , this can be done in  $\binom{|A \triangle B|}{k_3} = \binom{2n-2-2c_{AB}-2c_{ABC}}{k_3}$  ways.

STEP 4: Choose the edge list of  $(A, B, C)$ .

As all edges of  $C$  that are common with  $A$  or  $B$  are already in the edge list of  $(A, B)$ , it only remains to insert  $|C \setminus (A \cup B)| = n-1-c_{AC}-c_{BC}-c_{ABC}$  many C:s into this list. There are at least  $c_{AC}+c_{BC}+c_{ABC}-k_2$  entries in the edge list of  $(A, B)$  immediately after which we cannot place any C:s, namely those corresponding to edges in common segments between  $C$  and  $A \cup B$  that are not the last edge in its segment. Hence, the number of ways this extension can

be made is at most  $\binom{|A \cup B| + |C \setminus (A \cup B)| - (c_{AC} + c_{BC} + c_{ABC} - k_2)}{|C \setminus (A \cup B)|} = \binom{3n - 3 - c_{AB} - 2c_{AC} - 2c_{BC} - 3c_{ABC} + k_2}{n - 1 - c_{AC} - c_{BC} - c_{ABC}}.$

STEP 5: Choose the vertex sequences of  $A, B$  and  $C$ .

There are  $n!$  possibilities for the vertex sequence of  $A$ . Given  $G$  and the edge-ordering of  $\{A, B, C\}$  this determines  $c_{AB} + c_{ABC} + k_1 + k_4$  of the vertices along  $B$ , yielding at most  $(n - c_{AB} - c_{ABC} - k_1 - k_4)!$  options for the remaining vertices of  $B$ . Similarly, fixing  $A$  and  $B$ , the remaining vertices along  $C$  can be chosen in at most  $(n - c_{AC} - c_{BC} - c_{ABC} - k_2)!$  ways.  $\square$

**Lemma 3.6.** *For any non-negative integers  $p \leq q \leq r$ ,*

$$\frac{(r-p)!}{(q-p)!} \leq \frac{r!}{q!} \left(\frac{q}{r}\right)^p \leq \frac{r!}{q!} \exp\left(-\frac{p(r-q)}{r}\right),$$

and

$$\frac{(r-p)!}{r!} \leq \left(\frac{e}{r}\right)^p.$$

*Proof.* We have

$$\frac{q!}{(q-p)!} \frac{(r-p)!}{r!} = \frac{q-p+1}{r-p+1} \cdot \frac{q-p+2}{r-p+2} \cdots \frac{q}{r} \leq \left(\frac{q}{r}\right)^p,$$

where, by convexity of the exponential function,  $\frac{q}{r} = 1 - \frac{r-q}{r} \leq e^{-\frac{r-q}{r}}$ . Moreover,

$$\begin{aligned} \ln\left(\frac{(r-p)!}{r!}\right) &= \sum_{t=r-p+1}^r -\ln t \\ &\leq \int_{r-p}^r -\ln t \, dt = (r-p) \ln(r-p) - r \ln r + p \\ &\leq (r-p) \ln r - r \ln r + p = p(1 - \ln r). \end{aligned}$$

$\square$

We are now ready to derive an upper bound on  $\mathbb{E}X^3$  that shows that the quantity is of order  $n^3$  and further lets us identify the dominating contribution. We define a reduced common edge graph  $\tilde{G}$  as *good* if each of its components consists of a single edge which is common to precisely two paths.

**Proposition 3.7.** *We have  $\mathbb{E}X^3 = O(n^3)$ . Moreover, for any integer  $M \geq 0$ , the contribution to the sum*

$$\mathbb{E}X^3 = \sum_{\tilde{G}} \sum_{\bar{c}} t_n(\tilde{G}, \bar{c})$$

*from all pairs  $(\tilde{G}, \bar{c})$  such that either  $\tilde{G}$  is not good or  $c_{AB} + c_{AC} + c_{BC} > M$  is  $O(n^3)e^{-\Omega(M)} + O(n^2)$ .*

*Proof.* Fix  $\varepsilon > 0$  sufficiently small ( $\varepsilon \leq 1500000^{-1}$  suffices). By Claim 3.3 we know that we can bound  $\mathbb{E}X^3 = \sum_{(A,B,C, \preceq)} \frac{1}{|A \cup B \cup C|!}$  by  $3S_1 + 3S_2 + S_3$ , where  $S_1$  is the contribution to this sum from all edge-ordered triples where

$$|A \cap B| \leq (1 - \varepsilon)n \quad \text{and} \quad |(A \cup B) \cap C| \leq (1 - \varepsilon)n,$$

$S_2$  is the contribution where

$$|A \cap B| > (1 - \varepsilon)n \quad \text{and} \quad |(A \cup B) \cap C| \leq (1 - \varepsilon)n,$$

and  $S_3$  is the contribution where  $|A \cap B \cap C| > (1 - 18\varepsilon)n$ .

Let us start by estimating  $S_1$ . We have

$$S_1 = \sum_{\tilde{G}} \sum_{\substack{c_{AB}+c_{ABC} \leq (1-\varepsilon)n \\ c_{AC}+c_{BC}+c_{ABC} \leq (1-\varepsilon)n}} t_n(\tilde{G}, \bar{c}).$$

To bound the summand in the right-hand side we apply Proposition 3.4. Note that we can assume that  $c_{AB} + c_{ABC} - k_1 \geq 0$  and  $c_{AC} + c_{BC} + c_{ABC} - k_2 \geq 0$ , as otherwise  $t_n(\tilde{G}, \bar{c}) = 0$ . By the first part of Lemma 3.6 using  $p = c_{AB} + c_{ABC} - k_1$ ,  $q = n - 1 - k_1$  and  $r = 2n - 2 - c_{AB} - c_{ABC}$  we have, for any  $c_{AB}, c_{AC}, c_{BC}, c_{ABC}$  as in the above sum,

$$\begin{aligned} & \binom{2n - 2 - 2c_{AB} - 2c_{ABC} + k_1}{n - 1 - c_{AB} - c_{ABC}, n - 1 - c_{AB} - c_{ABC}, k_1} \\ & \leq \binom{2n - 2 - c_{AB} - c_{ABC}}{n - 1 - c_{AB} - c_{ABC}, n - 1 - k_1, k_1} e^{-\Omega_\varepsilon(c_{AB} + c_{ABC} - k_1)} \end{aligned} \quad (3.5)$$

and using  $p = c_{AC} + c_{BC} + c_{ABC} - k_2$ ,  $q = 2n - 2 - c_{AB} - c_{ABC}$  and  $r = 3n - 3 - c_{AB} - c_{AC} - c_{BC} - 2c_{ABC}$

$$\begin{aligned} & \binom{3n - 3 - c_{AB} - 2c_{AC} - 2c_{BC} - 3c_{ABC} + k_2}{n - 1 - c_{AC} - c_{BC} - c_{ABC}} \\ & \leq \binom{3n - 3 - c_{AB} - c_{AC} - c_{BC} - 2c_{ABC}}{n - 1 - c_{AC} - c_{BC} - c_{ABC}} e^{-\Omega_\varepsilon(c_{AC} + c_{BC} + c_{ABC} - k_2)}. \end{aligned} \quad (3.6)$$

Plugging this into Proposition 3.4 we get, after some simplification,

$$\begin{aligned} t_n(\tilde{G}, \bar{c}) & \leq \left( \prod_{\alpha} \binom{c_{\alpha} - 1}{l_{\alpha} - 1} e^{-\Omega_\varepsilon(c_{\alpha})} \right) \frac{e^{O_\varepsilon(k_1 + k_2 + k_3)} n^{k_3}}{k_1! k_3!} \frac{n!}{(n - 1 - k_1)!} \\ & \quad \cdot \frac{(n - c_{AB} - c_{ABC} - k_1 - k_4)! (n - c_{AC} - c_{BC} - c_{ABC} - k_2)!}{(n - 1 - c_{AB} - c_{ABC})! (n - 1 - c_{AC} - c_{BC} - c_{ABC})!} \\ & \leq O(n^3) \left( \prod_{\alpha} \binom{c_{\alpha} - 1}{l_{\alpha} - 1} e^{-\Omega_\varepsilon(c_{\alpha})} \right) \frac{e^{O_\varepsilon(k_1 + k_2 + k_3 + k_4)}}{k_1! k_3! n^{k_2 + k_4 - k_3}}, \end{aligned}$$

where in the last step we used  $\frac{n!}{(n - 1 - k_1)!} \leq n^{k_1 + 1}$  and, by the second part of Lemma 3.6,  $\frac{(n - c_{AB} - c_{ABC} - k_1 - k_4)!}{(n - c_{AB} - c_{ABC})!} \leq n^{-k_1 - k_2} e^{O_\varepsilon(k_1 + k_4)}$  and  $\frac{(n - c_{AC} - c_{BC} - c_{ABC} - k_2)!}{(n - c_{AC} - c_{BC} - c_{ABC})!} \leq n^{-k_2} e^{O_\varepsilon(k_2)}$ .

Using the Taylor expansion  $\frac{z^l}{(1 - z)^l} = \sum_c \binom{c - 1}{l - 1} z^c$  it follows that  $\sum_{c_{\alpha}} \binom{c_{\alpha} - 1}{l_{\alpha} - 1} e^{-\Omega_\varepsilon(c_{\alpha})} \leq e^{O_\varepsilon(l_{\alpha})}$ . By the second part of Claim 3.2 we know that  $\sum_{\alpha} l_{\alpha} = O(k_1 + k_2 + k_4)$ . Hence

$$S_1 \leq O(n^3) \sum_{\tilde{G}} \frac{e^{O_\varepsilon(k_1 + k_2 + k_3 + k_4)}}{k_1! k_3! n^{k_2 + k_4 - k_3}}. \quad (3.7)$$

By the definitions of  $k_2, k_3$  and  $k_4$ , equations (3.2)-(3.4), one can see that  $k_2 + k_4 - k_3$  is at least the number of components of  $\tilde{G}$  that either consist of more than one edge, or contain an  $ABC$ -edge. In particular,  $k_2 + k_4 - k_3$  is non-negative, and zero only if  $\tilde{G}$  is good. Using this observation together with the first part of Claim 3.2, it is easy to see that the sum in (3.7) converges uniformly for sufficiently large  $n$ . Hence  $S_1 = O_\varepsilon(n^3)$ , and moreover the contribution from non-good  $\tilde{G}$  is  $O_\varepsilon(n^2)$ .

We now turn to  $S_2$ . Similarly to above we have

$$S_2 = \sum_{\tilde{G}} \sum_{\substack{c_{AB}+c_{ABC}>(1-\varepsilon)n \\ c_{AC}+c_{BC}+c_{ABC}\leq(1-\varepsilon)n}} t_n(\tilde{G}, \bar{c}).$$

Let  $d_{AB} = n - c_{AB} - c_{ABC}$ . Note that  $d_{AB} \geq 1$ . Again, applying Proposition 3.4, and simplifying using (3.6),  $\binom{2d_{AB}-2+k_1}{d_{AB}-1, d_{AB}-1, k_1} \leq 3^{2d_{AB}-2+k_1}$  and  $\binom{2d_{AB}-2}{k_3} \leq 2^{2d_{AB}-2}$ , we get, after some cancellation,

$$\begin{aligned} t_n(\tilde{G}, \bar{c}) &\leq \frac{n^{l_{AB}-1}}{(l_{AB}-1)!} \left( \prod_{\alpha \neq AB} \binom{c_\alpha-1}{l_\alpha-1} e^{-\Omega_\varepsilon(c_\alpha)} \right) 36^{d_{AB}-1} e^{O_\varepsilon(k_1+k_2)} \\ &\quad \cdot \frac{n!(d_{AB}-k_1-k_4)!}{(n+d_{AB}-2)!} n^{1-k_2} e^{O_\varepsilon(k_2)} \\ &\leq O(n^2) \left( \prod_{\alpha \neq AB} \binom{c_\alpha-1}{l_\alpha-1} e^{-\Omega_\varepsilon(c_\alpha)} \right) (36\varepsilon)^{d_{AB}} \\ &\quad \cdot \frac{e^{O_\varepsilon(k_1+k_2+k_4)}}{(l_{AB}-1)! n^{k_1+k_2+k_4-l_{AB}}}, \end{aligned}$$

where the last step uses  $\frac{n!(d_{AB}-k_1-k_4)!}{(n+d_{AB}-2)!} \leq \frac{d_{AB}^{d_{AB}-k_1-k_4}}{n^{d_{AB}-2}} \leq \varepsilon^{d_{AB}-k_1-k_4} n^{2-k_1-k_4}$ . Assuming  $\varepsilon < \frac{1}{36}$  it follows by summing over  $d_{AB}, c_{AC}, c_{BC}, c_{ABC}$  that

$$S_2 \leq O_\varepsilon(n^2) \sum_{\tilde{G}} \frac{e^{O_\varepsilon(k_1+k_2+k_4)}}{(l_{AB}-1)! n^{k_1+k_2+k_4-l_{AB}}}.$$

We will now prove that  $k_1 + k_2 \geq l_{AB}$ . Consider an edge  $e$  in  $\tilde{G}$  labelled with  $AB$ . Either it is the lowest ordered edge in a common segment of  $A$  and  $B$ , and hence counted in  $k_1$ , or there is a preceding edge  $e'$ . In the latter case, we know that  $e'$  is common to  $A$  and  $B$ , and since  $\tilde{G}$  contains no incident edges with the same label,  $e'$  must be labelled  $ABC$ . The claim then is that  $e'$  is the last edge in a common segment of  $A \cup B$  and  $C$ , which means that it is counted in  $k_2$ . If not, there is a next edge  $e''$  which, without loss of generality, is labelled  $AC$ . But then  $e, e'$  and  $e''$  are consecutive edges in  $A$  which are not ordered monotonely, which is a contradiction. Again, by the first part of Claim 3.2, this is summable yielding a contribution of  $O(n^2)$ .

Lastly, we have

$$S_3 = \sum_{\tilde{G}} \sum_{c_{ABC} > (1-18\varepsilon)n} t_n(\tilde{G}, \bar{c}).$$

Let  $d_{ABC} = n - c_{ABC}$ . For  $c_{ABC} > (1-18\varepsilon)n$  we get

$$\begin{aligned} t_n(\tilde{G}, \bar{c}) &\leq \frac{n^{l_{ABC}-1}}{(l_{ABC}-1)!} \left( \prod_{\alpha \neq ABC} \binom{c_\alpha-1}{l_\alpha-1} \right) 3^{2d_{ABC}-2-2c_{AB}+k_1} 2^{2d_{ABC}-2-2c_{AB}} \\ &\quad \cdot 2^{3d_{ABC}-3-c_{AB}-2c_{AC}-2c_{BC}+k_2} \frac{n!(d_{ABC}-c_{AB}-k_1-k_4)!(d_{ABC}-c_{AC}-c_{BC}-k_2)!}{(n+2d_{ABC}-3-c_{AB}-c_{AC}-c_{BC})!} \\ &\leq \frac{n^{l_{ABC}-1}}{(l_{ABC}-1)!} \left( \prod_{\alpha \neq ABC} \binom{c_\alpha-1}{l_\alpha-1} 4^{-c_\alpha} \right) 288^{d_{ABC}-1} \\ &\quad \cdot (18\varepsilon)^{2d_{ABC}-c_{AB}-c_{AC}-c_{BC}} O(n^{3-k_1-k_2-k_4}) e^{O_\varepsilon(k_1+k_2+k_4)}, \end{aligned}$$

where, as before,  $\frac{n!(d_{ABC}-c_{AB}-k_1-k_4)!(d_{ABC}-c_{AC}-c_{BC}-k_2)!}{(n+2d_{ABC}-3-c_{AB}-c_{AC}-c_{BC})!} \leq (18\varepsilon)^{2d_{ABC}-c_{AB}-c_{AC}-c_{BC}} n^{3-k_1-k_2-k_4}$ . Note that  $d_{ABC} - c_{AB} - c_{AC} - 1 \geq 0$  as this denotes the number of edges of  $A \setminus (B \cup C)$ , and similarly  $d_{ABC} - c_{AB} - c_{BC} - 1 \geq 0$  and  $d_{ABC} - c_{AC} - c_{BC} - 1 \geq 0$ . Hence  $3d_{ABC} - 2c_{AB} - 2c_{AC} - 2c_{BC} \geq 0$  and thus  $2d_{ABC} - c_{AB} - c_{AC} - c_{BC} \geq \frac{d_{ABC}}{2}$ . It follows that, for  $\varepsilon \leq \frac{1}{18}$ ,

$$t_n(\tilde{G}, \bar{c}) \leq O(n^2) \left( \prod_{\alpha \neq ABC} \binom{c_\alpha - 1}{l_\alpha - 1} 4^{-c_\alpha} \right) (288\sqrt{18\varepsilon})^{d_{ABC}} \cdot \frac{e^{O_\varepsilon(k_1+k_2+k_4)}}{(l_{ABC} - 1)! n^{k_1+k_2+k_4-l_{ABC}}},$$

and thus, assuming  $\varepsilon < \frac{1}{288^2 \cdot 18}$ , we have

$$S_3 \leq O_\varepsilon(n^2) \sum_{\tilde{G}} \frac{e^{O_\varepsilon(k_1+k_2+k_4)}}{(l_{ABC} - 1)! n^{k_1+k_2+k_4-l_{ABC}}}.$$

To show that this sum over  $\tilde{G}$  is bounded the key observation is that  $l_{ABC} \leq k_1 + k_2$  for any  $\tilde{G}$ , as each  $ABC$ -edge in  $\tilde{G}$  is either the first edge in a common segment of  $A$  and  $B$ , or the first edge in a common segment of  $A \cup B$  and  $C$ . We show this through proof by contradiction. Let  $e$  be an  $ABC$ -edge in  $\tilde{G}$  not of this form, let  $e'$  be the preceding edge in the  $AB$ -component, and  $e''$  the preceding edge in the  $C$ -component. Then, as no two incident edges in  $\tilde{G}$  have the same label,  $e'$  must be labelled  $AB$  and  $e''$  must either be labelled  $AC$  or  $BC$ , say, without loss of generality,  $AC$ . Then both  $e'$  and  $e''$  incident to  $e$  in the edge-ordering of  $A$ , but they are both before  $e$  in the ordering, which is impossible. Using this it follows that  $S_3 = O_\varepsilon(n^2)$ . We remark that actually  $l_{ABC} < k_1 + k_2$ , which strengthens the bound to  $S_3 = O_\varepsilon(n)$ , though this is not required here.

From the preceding argument we know that the contribution to  $\mathbb{E}X^3$  from non-good reduced common edge graphs, and from  $\bar{c}$  such that  $c_{AB} + c_{AC} + c_{BC} > M$  is at most

$$O_\varepsilon(n^2) + 3 \sum_{\tilde{G}} \sum_{\substack{c_{AB}+c_{ABC} \leq (1-\varepsilon)n \\ c_{AC}+c_{BC}+c_{ABC} \leq (1-\varepsilon)n \\ c_{AB}+c_{AC}+c_{BC} > M}} t_n(\tilde{G}, \bar{c}). \quad (3.8)$$

Using the bound

$$t_n(\tilde{G}, \bar{c}) \leq n^3 e^{-\Omega_\varepsilon(M/2)} \left( \prod_{\alpha} \binom{c_\alpha - 1}{l_\alpha - 1} e^{-\Omega_\varepsilon(c_\alpha/2)} \right) \frac{e^{O_\varepsilon(k_1+k_2+k_3+k_4)}}{k_3! n^{k_2+k_4-k_3}},$$

which is valid for any term in the sum in (3.8), and proceeding as before with the bound of  $S_1$  it follows that the above sum is  $O(n^3) e^{-\Omega_\varepsilon(M/2)}$ , as desired.  $\square$

It remains to investigate the contribution from good reduced common edge graphs and small  $\bar{c}$  more carefully. Recall that a good reduced common edge graph  $\tilde{G}$ , by definition, has no edges common to all three paths. Hence  $t_n(\tilde{G}, c_{AB}, c_{AC}, c_{BC}, c_{ABC}) = 0$  unless  $c_{ABC} = 0$ .

**Proposition 3.8.** *Let  $\tilde{G}$  be a fixed good reduced common edge graph, and let  $c_{AB}, c_{AC}$  and  $c_{BC}$  be fixed non-negative integers. Then,*

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{-3} t_n(\tilde{G}, c_{AB}, c_{AC}, c_{BC}, 0) \\ &= \frac{e^{-6}}{(\sum_{\alpha} k_{\alpha})!} \prod_{\alpha} \left( \binom{c_{\alpha} - r_{\alpha} - 1}{k_{\alpha} - r_{\alpha} - 1} 2^{-c_{\alpha} + k_{\alpha}} \right), \end{aligned}$$

where the the sum and product over  $\alpha$  go over  $AB, AC$ , and  $BC$ ,  $k_\alpha = k_\alpha(\tilde{G})$  denotes the number of components in  $\tilde{G}$  labelled with  $\alpha$ , and  $r_\alpha = r_\alpha(\tilde{G})$  denotes the number of these where the paths traverse the segment in opposite directions.

*Proof.* Given an edge-ordered triple  $A, B, C$ , we say that an edge  $e \in A \cup B \cup C$  occurs during an edge set  $E \subseteq A \cup B \cup C$  if there exists  $e', e'' \in E$  such that  $e' \preceq e \preceq e''$ . Thus, for an edge-ordered triple  $A, B, C$ , we can assign numbers  $m_{AB}, m_{AC}$  and  $m_{BC}$  counting the number of unique edges of  $C$ ,  $B$  or  $A$  respectively that occur during some common segment of  $A$  and  $B$ ,  $A$  and  $C$  or  $B$  and  $C$ .

We now count the number of edge-ordered triples corresponding to fixed  $\tilde{G}$ ,  $c_{AB}, c_{AC}, c_{BC}$  and given values of  $m_{AB}, m_{AC}$  and  $m_{BC}$ . Since each common segment of an edge-ordered triple contains at least one edge, we may assume that  $c_\alpha \geq k_\alpha = k_\alpha(\tilde{G})$  for  $\alpha = AB, AC, BC$ .

STEP 1: Choose a common edge graph  $G$ .

For each  $\alpha \in \{AB, AC, BC\}$  we need to choose the length of each of the  $k_\alpha - r_\alpha$  segments that the paths traverse in the same direction. This is equal to the number of ways to place  $c_\alpha - r_\alpha$  indistinguishable balls into  $k_\alpha - r_\alpha$  bins such that each bin contains at least one ball. Hence the number of ways to do this is  $\prod_\alpha \binom{c_\alpha - r_\alpha - 1}{k_\alpha - r_\alpha - 1}$ .

STEP 2: Choose an edge list of  $(A, B, C)$ .

Let us first consider the edges unique to one of the paths that occur during a common segment of the other two. For a given  $\alpha$ , there are  $c_\alpha - k_\alpha$  spaces between common edges of  $\alpha$ . Hence the number of ways to place these  $m_\alpha$  edges is equal to the number of ways to place  $m_\alpha$  indistinguishable balls into  $c_\alpha - k_\alpha$  bins, which is  $\binom{m_\alpha + c_\alpha - k_\alpha - 1}{c_\alpha - k_\alpha - 1}$ .

It remains to choose the placement of the remaining unique edges of each path. This is equivalent to intertwining four strings: one consisting of  $n - 1 - m_{BC} - c_{AB} - c_{AC}$  A:s, one of  $n - 1 - m_{AC} - c_{AB} - c_{BC}$  B:s, one of  $n - 1 - m_{AB} - c_{AC} - c_{BC}$  C:s, and one of  $\sum_\alpha k_\alpha$  D:s, where the D:s are placeholders for the common segments. This can be done in

$$\frac{(3n - 3 - \sum_\alpha (2c_\alpha + m_\alpha - k_\alpha))!}{(\sum_\alpha k_\alpha)! \prod_\alpha (n - 1 - m_\alpha - \sum_{\alpha' \neq \alpha} c_{\alpha'})!}$$

ways. However, this will overestimate the number of edge lists, for instance, as  $\tilde{G}$  is good we cannot have two D:s next to each other. The exact condition on such an intertwined string to yield a feasible edge list is a bit involved to describe, but as we shall see, it is sufficient that the string contains at least two A:s, two B:s and C:s between each pair of D:s.

To estimate the proportion of such intertwined strings, consider a random such string, chosen with uniform probability. Observe that the subsequence consisting of all A:s and D:s is uniformly distributed among the possible strings consisting of  $n - 1 - m_{BC} - c_{AB} - c_{AC}$  A:s and  $\sum_\alpha k_\alpha$  D:s. Hence, by the first moment method, the probability that two D:s have less than two A:s between them is  $o_{\bar{c}, \bar{m}}(1)$  where  $\bar{m} = (m_{AB}, m_{AC}, m_{BC})$ . The argument works analogously for the subsequences consisting of all B:s and D:s or C:s and D:s.

STEP 3: Choose the vertex sequences of  $A, B$  and  $C$ .

Note that given  $G$  and the edge list of  $(A, B, C)$  we can determine which vertices along the respective paths that are contained in common segments with one of the other two paths, and thereby which vertices in  $G$  these correspond to.

Suppose that we first choose an embedding of  $G$  into  $K_n$ . This can be done in  $\frac{n!}{(n - \sum_{\alpha} (c_{\alpha} + k_{\alpha}))!} \sim n^{\sum_{\alpha} c_{\alpha} + k_{\alpha}}$  ways. This determines the vertices at  $c_{AB} + c_{AC} + k_{AB} + k_{AC}$  positions along  $A$ ,  $c_{AB} + c_{BC} + k_{AB} + k_{BC}$  ones along  $B$  and  $c_{AC} + c_{BC} + k_{AC} + k_{BC}$  ones along  $C$ . Second, we assign the remaining parts of the vertex sequences of  $A$ ,  $B$  and  $C$  such that the paths are Hamiltonian, which can be done in  $\prod_{\alpha} (n - \sum_{\alpha' \neq \alpha} (c_{\alpha'} + k_{\alpha'}))! \sim n!^3 n^{-2 \sum_{\alpha} (c_{\alpha} + k_{\alpha})}$  ways.

Again, not all of these assignments are allowed as it may be the case that the vertex sequences gives rise to more common edges than those in  $G$ .

**Claim 3.9.** *Assume the edge list of  $(A, B, C)$  contains two  $A$ :s, two  $B$ :s and two  $C$ :s between each pair of common segments. Then, the proportion of vertex sequences as above that yield no additional overlap between the paths is  $e^{-6} \pm o_{\bar{c}}(1)$ .*

Let us postpone the proof of this claim for now. By the preceding argument it follows that for any fixed  $\tilde{G}$  and  $c_{AB}, c_{AC}$  and  $c_{BC}$ , we have

$$\begin{aligned} & n^{-3} t_n(\tilde{G}, c_{AB}, c_{AC}, c_{BC}, 0) \\ & \sim \sum_{\bar{m}} \frac{e^{-6} n^{-3}}{(\sum_{\alpha} k_{\alpha})!} \left( \prod_{\alpha} \binom{c_{\alpha} - r_{\alpha} - 1}{k_{\alpha} - r_{\alpha} - 1} \binom{m_{\alpha} + c_{\alpha} - k_{\alpha} - 1}{c_{\alpha} - k_{\alpha} - 1} \right) \\ & \quad \cdot \frac{(3n - 3 - (\sum_{\alpha} 2c_{\alpha} + m_{\alpha} - k_{\alpha}))! n!^3 n^{-\sum_{\alpha} (c_{\alpha} + k_{\alpha})}}{(3n - 3 - \sum_{\alpha} c_{\alpha})! \prod_{\alpha} (n - 1 - m_{\alpha} - \sum_{\alpha' \neq \alpha} c_{\alpha'})!} (1 - o_{\bar{c}, \bar{m}}(1)). \end{aligned} \quad (3.9)$$

Using the bounds

$$\frac{n!}{(n - 1 - m_{\alpha} - \sum_{\alpha' \neq \alpha} c_{\alpha'})!} \leq n^{1 + m_{\alpha} + \sum_{\alpha' \neq \alpha} c_{\alpha'}}$$

and

$$\frac{(3n - 3 - (\sum_{\alpha} 2c_{\alpha} + m_{\alpha} - k_{\alpha}))!}{(3n - 3 - \sum_{\alpha} c_{\alpha})!} \leq \left( \frac{e}{3n(1 - o_{\bar{c}}(1))} \right)^{\sum_{\alpha} c_{\alpha} + m_{\alpha} - k_{\alpha}},$$

where the latter follows from the second part of Lemma 3.6, we see that the summand in (3.9) decreases exponentially in  $m_{AB}, m_{AC}$  and  $m_{BC}$  uniformly over  $n$ . Hence, taking the term-wise limit of (3.9) it follows by dominated convergence that

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{-3} t_n(\tilde{G}, c_{AB}, c_{AC}, c_{BC}, 0) \\ & = \frac{e^{-6}}{(\sum_{\alpha} k_{\alpha})!} \prod_{\alpha} \left( \binom{c_{\alpha} - r_{\alpha} - 1}{k_{\alpha} - r_{\alpha} - 1} \sum_{m_{\alpha}} \binom{m_{\alpha} + c_{\alpha} - k_{\alpha} - 1}{c_{\alpha} - k_{\alpha} - 1} 3^{-m_{\alpha} - c_{\alpha} + k_{\alpha}} \right) \\ & = \frac{e^{-6}}{(\sum_{\alpha} k_{\alpha})!} \prod_{\alpha} \left( \binom{c_{\alpha} - r_{\alpha} - 1}{k_{\alpha} - r_{\alpha} - 1} 2^{-c_{\alpha} + k_{\alpha}} \right), \end{aligned}$$

where in the last step we again used  $\frac{z^l}{(1-z)^l} = \sum_c \binom{c-1}{l-1} z^c$  with  $z = \frac{1}{3}$ ,  $l = c_{\alpha} - k_{\alpha}$  and  $c = m_{\alpha} + c_{\alpha} - k_{\alpha}$ .  $\square$

*Proof of Claim 3.9.* Here we make use of Brun's sieve, see e.g. Theorem 8.3.1 in [2]: Let  $\xi = \xi_n$  be a sequence of non-negative integer-valued random variables. Suppose there exists a constant  $\mu$  such that  $\mathbb{E} \xi_n \rightarrow \mu$  and moreover  $\mathbb{E} \binom{\xi_n}{r} \rightarrow \frac{\mu^r}{r!}$  for all positive integers  $r$  as  $n \rightarrow \infty$ . Then  $\mathbb{P}(\xi_n = r) \rightarrow \frac{\mu^r}{r!} e^{-\mu}$  for any  $r = 0, 1, \dots$

Consider a random assignment of vertex sequences for  $A, B$  and  $C$  chosen with probability among the assignments as described in the proof of Proposition 3.8. Let  $R$  denote the range of the embedding of  $G$  into  $K_n$ . We first observe that the expected number of coinciding edges



where at least one end-point is in  $R$  is  $o_{\bar{c}}(1)$ . Hence it remains to consider the case of coinciding edges where neither end-point is in  $R$ .

Condition on  $R$  and the vertex sequence of  $A$ . Let  $e_0 = \{u_0, v_0\}, e_1 = \{u_1, v_1\}, \dots$  be an indexing of the edges of  $A$  that have no end-points in  $R$ , and for each  $i$ , let  $E_{e_i}$  and  $F_{e_i}$  be the events that  $u_i v_i$  respectively  $v_i u_i$  are substrings of  $B$ . Let  $\xi = \sum_i \mathbb{1}_{E_{e_i}} + \mathbb{1}_{F_{e_i}}$ . Then  $\mathbb{E}(\xi_r) = \sum_{|\mathcal{E}|=r} \mathbb{P}(\cap_{E \in \mathcal{E}} E)$  where the sum goes over all unordered  $r$ -tuples of events  $E_{e_i}$  and  $F_{e_i}$  as above.

The events in an  $r$ -tuple  $\mathcal{E}$  may be incompatible - there might be two events saying that  $B$  should contain substrings starting or ending with the same vertex, in which case clearly  $\mathbb{P}(\cap_{E \in \mathcal{E}} E) = 0$ . Otherwise  $\cap_{E \in \mathcal{E}} E$  is the event that  $B$  contains substrings  $\{w_0, w_1, \dots, w_{r_0}\}, \{w'_0, w'_1, \dots, w'_{r_1}\}, \dots$  and so on where  $\sum_i r_i = r$ . Again, for fixed  $c_{AB}, c_{AC}$  and  $c_{BC}$  it is straight-forward to see that the number of compatible  $r$ -tuples is  $\sim \frac{(2n)^r}{r!}$ , and for any such  $\mathcal{E}$ ,  $\mathbb{P}(\cap_{E \in \mathcal{E}} E) \sim n^{-r}$ . Hence, by Brun's sieve, the probability that  $\xi = 0$ , that is, that there are no extra common edges between  $A$  and  $B$ , is  $\sim e^{-2}$ .

Now, condition on  $R$  and the vertex sequences of  $A$  and  $B$ . Repeating the argument above, but now counting the number of coinciding edges between  $A \cup B$  and  $C$ , which results in  $\sim \frac{(4n)^r}{r!}$  possible compatible  $r$ -tuples  $\mathcal{E}$ , it follows that the probability of no extra overlap is  $\sim e^{-4}$ .  $\square$

*Proof of Proposition 2.3.* Note that for fixed  $\bar{c}$  only a finite number of  $\tilde{G}$  (up to isomorphism) yield non-zero  $t_n(\tilde{G}, \bar{c})$ . Hence, by Proposition 3.7 we have the term-wise limit

$$\lim_{n \rightarrow \infty} n^{-3} \mathbb{E} X^3 = \sum_{\tilde{G} \text{ good}} \sum_{\bar{c}} \lim_{n \rightarrow \infty} n^{-3} t_n(\tilde{G}, \bar{c}).$$

By Proposition 3.8 we thus have

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-3} \mathbb{E} X^3 &= \sum_{\tilde{G} \text{ good}} \frac{e^{-6}}{\left(\sum_{\alpha} k_{\alpha}(\tilde{G})\right)!} \\ &\cdot \prod_{\alpha} \left( \sum_{c_{\alpha}=0}^{\infty} \binom{c_{\alpha} - r_{\alpha}(\tilde{G}) - 1}{k_{\alpha}(\tilde{G}) - r_{\alpha}(\tilde{G}) - 1} 2^{-c_{\alpha} + r_{\alpha}(\tilde{G})} \cdot 2^{k_{\alpha}(\tilde{G}) - r_{\alpha}(\tilde{G})} \right) \\ &= \sum_{\tilde{G} \text{ good}} \frac{e^{-6}}{\left(\sum_{\alpha} k_{\alpha}(\tilde{G})\right)!} \prod_{\alpha} 2^{k_{\alpha}(\tilde{G}) - r_{\alpha}(\tilde{G})}. \end{aligned}$$

Yet again we used  $\frac{z^l}{(1-z)^l} = \sum_c \binom{c-1}{l-1} z^c$ , here with  $z = \frac{1}{2}$ ,  $l = k_{\alpha}(\tilde{G}) - r_{\alpha}(\tilde{G})$ ,  $c = c_{\alpha} - r_{\alpha}(\tilde{G})$ . Note that for given  $k_{AB}, k_{AC}, k_{BC}$  and  $r_{AB}, r_{AC}, r_{BC}$  the number of corresponding good reduced common edge graphs is  $\binom{k_{AB} + k_{AC} + k_{BC}}{k_{AB}, k_{AC}, k_{BC}} \prod_{\alpha} \binom{k_{\alpha}}{r_{\alpha}}$ . Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-3} \mathbb{E} X^3 &= e^{-6} \prod_{\alpha} \sum_{k_{\alpha}=0}^{\infty} \frac{2^{k_{\alpha}}}{k_{\alpha}!} \sum_{r_{\alpha}=0}^{k_{\alpha}} \binom{k_{\alpha}}{r_{\alpha}} 2^{-r_{\alpha}} \\ &= e^{-6} \prod_{\alpha} \sum_{k_{\alpha}=0}^{\infty} \frac{3^{k_{\alpha}}}{k_{\alpha}!} \\ &= e^3. \end{aligned}$$

The estimates for (2.4) and (2.5) are done analogously, but only count the contributions from terms where  $k_{AC} = 0$  and  $k_{AC} = k_{BC} = 0$  respectively.  $\square$

## Acknowledgements

I want to thank Klas Markström for introducing me to this problem, and my supervisor, Peter Hegarty, for his valuable input on an earlier draft.

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